Université Nice-Sophia Antipolis
Probability computational methods, 2024-2025

Exercices (1)

Exercise 1. (*Conditional distribution*) Let $X : (\Omega, \mathcal{A}, \mathbb{P}) \to (\mathbb{R}, \mathrm{Bor}(\mathbb{R}))$ be a real-valued random variable with cumulative distribution function F and left continuous canonical inverse[1] F_l^{-1}. Let $I = [a, b]$, $-\infty \leq a < b \leq +\infty$, be a nontrivial interval of $\mathbb{R}$. Show that, if $U \sim \mathcal{U}([0,1])$, then

$$F_l^{-1}(F(a) + (F(b) - F(a))U) \sim \mathcal{L}(X|X \in I)\,.$$

Exercise 2. (*The $\gamma(\alpha)$-distribution*) Let α be in $(0,1)$ and $\mathbb{P}_X(dx) = f_\alpha(x)\frac{dx}{\Gamma(\alpha)}$ where

$$f_\alpha(x) = x^{\alpha-1}e^{-x}\mathbb{1}_{\{x>0\}}(x)\,.$$

(Keep in mind that $\Gamma(a) = \int_0^{+\infty} u^{a-1}e^{-u}du$, $a > 0$). Note that when $\alpha = 1$, the gamma distribution is simply the exponential distribution. We will consider $E = (0, +\infty)$ and the reference σ-finite measure $\mu = \lambda_{/(0,+\infty)}$ (λ is the Lebesgue measure).

(1) Prove that

$$g_\alpha(x) = \frac{\alpha e}{\alpha + e}\left(x^{\alpha-1}\mathbb{1}_{\{0<x<1\}} + e^{-x}\mathbb{1}_{\{x\geq 1\}}\right)$$

is a probability density function.

(2) Show that $f_\alpha(x) < c_\alpha g_\alpha(x)$ for every x in $\mathbb{R}_+$, where

$$c_\alpha = \frac{\alpha + e}{\alpha e}\,.$$

(3) Let Y be a random variable of law $\mathbb{P}_Y(dy) = g_\alpha(y)\lambda(dy)$. Check that the cumulative distribution function of Y is

$$G_\alpha(x) = \frac{e}{\alpha + e}x^\alpha \mathbb{1}_{\{0<x<1\}} + \frac{\alpha e}{\alpha + e}\left(\frac{1}{e} + \frac{1}{\alpha} - e^{-x}\right)\mathbb{1}_{\{x>1\}}$$

and that, for every u in $(0,1)$,

$$G_\alpha^{-1}(u) = \left(\frac{\alpha + e}{e} \times u\right)^{\frac{1}{\alpha}}\mathbb{1}_{\left\{u<\frac{e}{\alpha+e}\right\}} - \log\left((1-u) \times \frac{\alpha + e}{\alpha e}\right)\mathbb{1}_{\left\{u\geq\frac{e}{\alpha+e}\right\}}.$$

(4) Try a `python` simulation (tips at the end of the sheet).
   (a) Code the functions f, g, G^{-1}.
   (b) Code a function returning a random variable of law $\mathbb{P}_X$ using the accept-reject method.
   (c) For $n = 10000$, draw a histogram of n variables drawn with the above method and a line representing the true density of these variables.

Exercise 3. Let $Z = (Z_1, Z_2)$ be a Gaussian vector such that Z_1 and Z_2 have the law $\mathcal{N}(0,1)$ and $\mathrm{Cov}(Z_1, Z_2) = \rho \in [-1, 1]$.

(1) Compute for every $u \in \mathbb{R}^2$, the Laplace transform $L(u) = \mathbb{E}(e^{\langle u, Z\rangle})$ ($\langle ., .\rangle$ is the scalar product).

(2) Compute for every σ_1, $\sigma_2 > 0$ the correlation $\rho_{X_1,X_2}(= \mathrm{Cov}(X_1, X_2)/\sqrt{\mathrm{Var}(X_1)\mathrm{Var}(X_2)})$ between the random variables $X_1 = e^{\sigma_1 Z_1}$, $X_2 = e^{\sigma_2 Z_2}$.

(3) Show that $\inf_{\rho\in[-1,1]} \rho_{X_1,X_2} \in (-1, 0)$ and that, when $\sigma_1 = \sigma_2 = \sigma$, $\inf_{\rho\in[-1,1]} \rho_{X_1,X_2} = e^{-\sigma^2}$.

PYTHON TIPS

Start by importing packages

[1] $\forall u \in [0, 1]\,,\ F_l^{-1}(u) = \inf\{x\ :\ F(x) \geq u\}$

```
import numpy as np
import scipy.stats as sps
import matplotlib.pyplot as plt
import math
```

- Function exp :

  ```
  print(np.exp(1))
  ```

- Uniform random variable:

  ```
  sps.uniform.rvs(size=1,loc=0,scale=1)
  ```

  see also `https://www.geeksforgeeks.org/python-uniform-distribution-in-statistics/`

- Array of zeros (indexed from 0 to 99):

  ```
  n=100
  E=np.zeros(n)
  ```

- Slice an interval ($[0, 5]$ here) into 400 steps and plot f:

  ```
  x=np.linspace(0,5,400)
  plt.plot(x,f(x))
  ```

- Γ function:

  ```
  print(math.gamma(1))
  ```

- Histogram (50 bins) renormalised to be a density:

  ```
  plt.hist(E,bins=50,density=True)
  ```

- Show your plots:

  ```
  plt.show()
  ```